value of the function when $x = 1$, that is, the sum of the coefficients of the function. As the arms are turned from 0° to 90°, keeping the slides properly adjusted and all the cords taut, the distance of this slide, R, from its initial position will be continuously the value of the function as x varies continuously from 1 to 0.

ON THE ORTHOCENTRIC QUADRILATERAL.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

Introduction. (*a*) The altitudes AD, BE, CF of a triangle ABC meets in a point H, the orthocenter of ABC. The triangle DEF formed by the feet D, E, F, of the altitudes is frequently called the orthic triangle of ABC.

To Carnot[2] is due the credit for having called attention to the almost obvious fact that *each of the four points H, A, B, C, is the orthocenter of the triangle formed by the other three.*

The points A, B, C, H are referred to as an *orthocentric group of points*, or an *orthocentric quadrilateral,* and the four triangles determined by these four points as an *orthocentric group of triangles.*

(*b*) In 1821 Brianchon and Poncelet showed that the circumcircle (N) of the orthic triangle DEF of ABC passes through the mid-points A', B', C', of the sides BC, CA, AB, of ABC, and also through the mid-points P, Q, R, of the segments AH, BH, CH respectively.[3] That the circle through the first six points mentioned passes also through the last three becomes obvious if we observe that *DEF is the orthic triangle not only of ABC, but of each of the four triangles of the orthocentric group $ABCH$.*

(*c*) In 1822 Feuerbach proved[4] that the circle (N) is tangent to the four circles which touch the sides of the triangle ABC. It was not until 1861 that Sir William R. Hamilton pointed out that (N) is also tangent to the circles touching the sides of the triangles BCH, CHA, HAB.[5] Now since the orthic triangle DEF is common to the four triangles of the orthocentric group $ABCH$, the circumcircle (N) of DEF is the nine-point circle of each of these four triangles, and therefore Hamilton's extension of Feuerbach's theorem becomes self-evident.

---

[1] Read before the American Mathematical Society, St. Louis, December 31, 1919. Readers of this article will be interested in comparing it with the first part of the author's earlier article "On the I-centre of a triangle" (*1918*, 241–246)—EDITOR.

[2] Carnot, *De la corrélation des figures de Géométrie*, 1801, p. 102.

[3] For the proof, see, for instance, J. Casey, *A Sequel to Euclid*, second edition, 1882, p. 58, or C. V. Durell, *Plane Geometry for Advanced Students*, vol. 1, pp. 30–31.

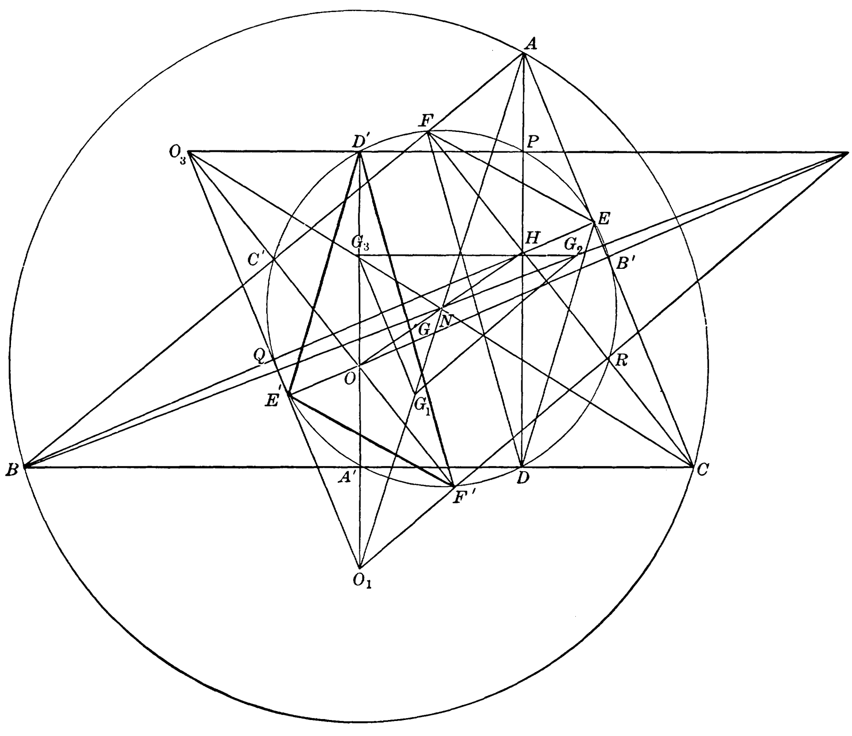
[4] For a proof see Casey, *l.c.*, pp. 58–61, or Durell, *l.c.*, pp. 46–47 and pp. 149–150.

[5] In making this statement Professor Altshiller-Court was possibly misled by Casey's reference to the result as "Sir William Hamilton's Theorem" (*Quarterly Journal of Mathematics*, 1861, p. 249) and by the fact that Sir William proposed the result as a problem in *Nouvelles annales de mathématiques*, 1861, vol. 20, p. 216.

The result was not, however, given originally by Sir William, but by T. T. Wilkinson, as prize-problem 1883 in *Lady's and Gentleman's Diary*, London, 1854, p. 72 (Solutions, *Diary*, 1855, pp. 67–69).—EDITOR.

These examples suggest that in certain connections it may be fruitful to consider the circle (N) as belonging not to the triangle ABC, but to the orthocentric group $ABCH$. The following considerations are based on this remark.

1. The center N of the nine-point circle (N) of the triangle ABC was shown by Benjamin Bevan, in 1804,[1] to lie midway between the orthocenter H and the circumcenter O of the triangle ABC. In other words, the circumcenter O of ABC is the symmetric of H with respect to N. But, as has been pointed out above, (N) is also the nine-point circle of the triangle BCH, whose orthocenter is the point A, hence the circumcenter O_1 of BCH is the symmetric of A with



respect to N. Similarly for the circumcenters O_2, O_3, of the triangles CHA, HAB. Consequently: *The four circumcenters of an orthocentric group of triangles form an orthocentric group which is the symmetric of the given group with respect to the nine-point center.*

2. From the symmetry of the two groups of points $ABCH$ and $O_1O_2O_3O$ follow immediately all the known properties of the circumcenters O_1, O_2, O_3, O. For instance:

(*a*) *The triangles $O_1O_2O_3$ and ABC are congruent*[2] and furthermore, *their sides are respectively parallel.* It may also be observed that these properties hold for the pairs of triangles O_2O_3O and BCH; O_3OO_1 and CHA; OO_1O_2 and HAB.

[1] For the proof compare Casey, or Durell, *l.c.*

[2] Durell, *l.c.*, p. 36, exercise 89.

(*b*) *The point H is the circumcenter of the triangle $O_1O_2O_3$.*[1] Similarly the points A, B, C, are the respective circumcenters of the triangles O_2O_3O, O_3OO_1, OO_1O_2.

(*c*) *The lines BO_3 and CO_2 are parallel*[1] and similarly for other pairs of analogous lines.

3. A wealth of other propositions, so far unobserved or unannounced, may be derived from the two symmetrical figures. We shall call attention to the following. In the symmetrical transformation considered the center of symmetry N is a double point, and the nine-point circle (N) is transformed into itself, hence: *An orthocentric group of triangles and the orthocentric group of their circumcenters have the same nine-point circle.*

4. The orthic triangle $D'E'F'$ of the orthocentric group $OO_1O_2O_3$ is the symmetric of the orthic triangle DEF of the orthocentric group $ABCH$, the pairs of points D, D'; E, E'; F, F'; being diametrically opposite on the circle (N). Thus we find a geometric interpretation of *three new points of the nine-point circle of the triangle ABC.*

5. The nine-point circle (N) of the orthocentric group $OO_1O_2O_3$ is tangent to the sixteen circles which touch the sides of the four triangles of this group, according to Hamilton's extension of Feuerbach's theorem (Introduction). These sixteen circles are the symmetric, with respect to N, of the analogous sixteen circles of the orthocentric group $ABCH$. Thus we find *sixteen new circles which are tangent to the nine-point circle of the triangle ABC.*[2]

6. The orthocentric group $OO_1O_2O_3$ has been derived by symmetry from the given orthocentric group $ABCH$. But the process may be reversed, and the group $HABC$ may be derived from the group $OO_1O_2O_3$ considered as given. Consequently: *The vertices of a given orthocentric group of triangles may be considered as the four circumcenters of a second orthocentric group of triangles, the two orthocentric groups having the same nine-point circle and being symmetrical with respect to its center.*

7. The point of intersection G of the medians of ABC, often referred to as the centroid of ABC, lies, according to a theorem of Euler,[3] on the line joining the orthocenter H to the circumcenter O of ABC, and we have $GO/GH = 1/2$. Since the nine-point center N is the midpoint of OH, we have $NG/NH = 1/3$, the points G and H being on opposite sides of N. In other words, the point G

---

[1] Durell, *l.c.*, p. 36, exercise 88.

[2] The results of paragraphs 1, 2, 3, 4 and 5 were given by T. T. Wilkinson in *Mathematical Questions with their Solutions from the Educational Times*, London, Vol. 1, 1864, pp. 6–7; see also *Mathematical Questions*, etc., Vol. 6, 1866, pp. 25–26.

T. T. Wilkinson seems to have been the first to discover an infinite series of circles tangent to the nine-point circle of a triangle (*Lady's and Gentleman's Diary*, London, 1857, p. 86; 1858, 87): "*If the radical centers of the inscribed and escribed circles of any triangle be taken, and circles be inscribed and escribed to the triangles formed by joining these radical centers, and the radical centers of the latter system of circles be again taken and circles inscribed and escribed to the triangles thus formed, and so on* ad infinitum, *the infinite number of circles thus found, as well as the original system of inscribed and escribed circles, always touch the circle drawn through the middle points of the* [sides of the] *first triangle.*"—EDITOR.

[3] Durell, *l.c.*, p. 41.

corresponds to H in a similitude of ratio $-1/3$, the center of similitude being N.[1] But N is also the nine-point center of the triangle BCH, whose orthocenter is A, hence the centroid G_1 of BCH corresponds to A in a similitude of ratio $-1/3$ with N as center of similitude. Similarly for the centroids G_2, G_3, of the triangles CHA, HAB. Consequently: *The four centroids of an orthocentric group of triangles form an orthocentric group, the two groups being similar and similarly placed.*

8. Since the centroids G, G_1, G_2, G_3, form an orthocentric group, all the properties of such a group immediately follow, as, for instance, that G is the orthocenter of the triangle $G_1G_2G_3$, etc.

Again the similitude of the two groups $GG_1G_2G_3$ and $HABC$ puts into evidence a great many properties, as for instance, that G_1G_2 is parallel to AB and is equal to 1/3 of its length; that the point of intersection of GG_1 and G_2G_3, which will be represented by (GG_1, G_2G_3), is collinear with N and $D \equiv (HA, BC)$; etc. The reader may find it interesting to formulate a number of these propositions.

9. In the similitude (7) by which the group $GG_1G_2G_3$ is derived from the group $HABC$, the center of similitude N is a double point. Hence: *An orthocentric group of triangles and the orthocentric group of their centroids have the same nine-point center.*

10. The orthocentric group $GG_1G_2G_3$ has been derived from the given orthocentric group $HABC$ by a similitude of center N and ratio $-1/3$. But the process may be reversed, and the orthocentric group $HABC$ may be derived from the orthocentric group $GG_1G_2G_3$, considered as given, by a similitude of ratio -3, the center remaining the same. Consequently: *The four points of an orthocentric group may be considered as the centroids of another orthocentric group of triangles, the two groups having the same nine-point center, this point being a center of similitude of the two groups, the ratio of similitude being -3.*

11. Since from (1) the two groups $HABC$ and $OO_1O_2O_3$ are symmetrical about the center N, therefore it follows from (10) that the two groups $GG_1G_2G_3$ and $OO_1O_2O_3$ admit N as a center of similitude, the ratio of similitude being $+3$. Hence: *The centroids and the circumcenters of an orthocentric group of triangles form two orthocentric groups of points having the same nine-point center, this point being a center of similitude of these two groups, the ratio of similitude being $+3$.*

1720

C. Maclaurin's *Geometria organica sive descriptio linearum curvarum universalis*, published at London—G. Poleni's *De mathesis in rebus physicis utilitate praelectio habita* ···, published at Patavia—Second edition of L'Hospital's *Traité analytiques des sections coniques*, published at Paris—Alexandre Savérien, author of *Dictionnaire universel de mathématiques et de physique* (2 vols., Paris, 1753), born July 16.

[1] Euler, *Novi comment. acad. sc. Petrop.*, vol. 11 (1765), 1767, p. 114.—Editor.